The Fundamentals of Linear Algebra

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Part I

What's Linear Algebra?

What's Linear Algebra?

- "Mathematics is the art of reducing any problem to linear algebra." -William Stein
- Linear algebra is one of the only mathematical theories that we understand almost completely
- All of math uses linear algebra as a source of examples, proof techniques, etc. Most problems are solved by reducing to linear algebra (e.g. linear approximation via derivatives)

SUMMER PLAN

- Review of the fundamentals (today): vector spaces, linear maps, bases, products, dual spaces
- Tensors: a unified language for multilinear maps
- Exterior Products: A powerful construction, using tensors, that allows intrinsic definitions of trace, determinant, and rank. Motivated geometrically by "area."
- Matrix operators: Power series of matrices (e.g. matrix exponential). Derivatives of these power series.
- Canonical forms: Diagonalizability and Jordan canonical form. Allows "reading off" the geometry of a matrix from the canonical form. Powerful for theory and applications.
- Scalar products: Generalizes the dot product in $\mathbb{R}^n$.
- Bilinear forms: Bilinear maps $V \times V \to \mathbb{R}$ (or $\mathbb{C}$).

Part II

VECTOR SPACES

Abstract Vector Spaces

Let k be a **field** (e.g. $\mathbb{R}$ or $\mathbb{C}$).

Definition 0.1

*A set V is a **vector space over** k if*

1. *V is an abelian group under operation $+$: there is a zero element 0, addition $u+v$ for $u, v \in V$ is defined, there are inverses $-u$ with $u+(-u)=0$, and $u+v=v+u$ (commutativity).*
2. *Scalar multiplication is defined: for $\lambda \in k$, $v \in V$, we have $\lambda v \in V$.*
3. *For all $u, v \in V$ and $\lambda, \mu \in k$,*

$$(\lambda+\mu)v = \lambda v + \mu v,\ \lambda(v+u) = \lambda v + \lambda u,\ 1v = v,\ 0v = 0$$

   (distributivity)

Note that the definition is abstract: we don't explain *how* addition and multiplication work, we just say what properties they must satisfy.

Examples of Vector Spaces

- $\mathbb{R}^n, \mathbb{C}^n, \mathbb{Q}^n$ are vector spaces over $\mathbb{R}, \mathbb{C}, \mathbb{Q}$ respectively.
- If $v \in \mathbb{R}^n$, the set $\{u \in \mathbb{R}^n : u \cdot v = 0\}$ is a vector space, the **orthogonal complement** ($\cdot$ is the usual dot product).
- The set of all real-valued continuous functions on $[0, 1]$, denoted $C([0, 1])$, is a vector space.
- So is $\{f \in C([0, 1]) : f(0) = f(1) = 0\}$.
- Polynomials of degree $\leq d$ over a field k form a finite-dimensional vector space over k.

Linear Independence and Bases

Definition 0.2

*A tuple of vectors $(v_1, \ldots, v_n)$ is **linearly dependent** if there exist $\lambda_1, \ldots, \lambda_n \in k$, not all equal to zero, such that*

$$\lambda_1 v_1 + \cdots + \lambda_n v_n = 0.$$

*If $(v_1, \ldots, v_n)$ are not linearly dependent, we say they are **linearly independent**.*

Definition 0.3

*A vector space is n**-dimensional** if there exists a linearly independent set of n vectors, but no linearly independent set of $n+1$ vectors. It is **infinite-dimensional** if there exist n linearly independent vectors for any n.*

- $\mathbb{R}^n$ is n-dimensional, but $C([a, b])$ is infinite-dimensional.

Definition 0.4 (Equivalent)

*The **dimension** of V is the longest length of any chain of strictly increasing vector subspaces*

$$V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n$$

Bases

Definition 0.5

*An **(ordered) basis** in a vector space V is a tuple $(e_1, \ldots, e_n)$ of linearly independent vectors such that any vector $v \in V$, can be expressed as $v = \sum_{k=1}^{n} v_k e_k$ for some $v_k \in k$ (i.e. $(e_1, \ldots, e_n)$ **span** V). The numbers v_k are the **components** of v with respect to $(e_1, \ldots, e_n)$.*

- This definition only works (as written) for finite-dimensional vector spaces. For infinite dimensions, you will have an infinite basis, but only *finite* linear combinations are allowed.
- A basis is *extra data* associated to V! There are many choices of basis for a given vector space.
- Equivalent definition: A basis is a **choice** of isomorphism from V to the "standard" vector space k^n, where $n = \dim V$. If $\varphi : V \to k^n$ is the isomorphism, the basis vectors are given by $e_i = \varphi^{-1}((0, \ldots, 0, 1, 0, \ldots, 0))$ with the 1 in the ith place.

Theorem 1

In a finite-dimensional vector space, all bases have equally many vectors (i.e. dimension is a well-defined integer).

Theorem 2

Every vector space has a basis.

This turns out to be equivalent to the axiom of choice! Think about what a basis of $\mathbb{R}$ over $\mathbb{Q}$ would look like.

Part III

LINEAR MAPS

LINEAR MAPS BETWEEN VECTOR SPACES

Definition 0.6

A function $A : V \to W$ between vector spaces V, W is ***linear*** *if for all $\lambda \in k$ and $u, v \in V$, $A(u + \lambda v) = A(u) + \lambda A(v)$.*

- If A is a linear map $V \to V$ and $(e_j)_j$ is a basis then there exist $A_{jk} \in k$ ($j, k = 1, \ldots, n$) such that if $v = \sum_{j=1}^n v_j e_j$, then $(Av)_j = \sum_{k=1}^n A_{jk} v_k$.
- By linearity, $Av = A(\sum_{k=1} v_k e_k) = \sum_{k=1} v_k A(e_k)$ so A is determined by where it sends e_k.
- In basis $(e_j)_j$, we can write Ae_k as $Ae_k = \sum_{j=1}^n A_j k e_j$.
- So $Av = \sum_{k=1}^n v_k \sum_{j=1}^n A_{jk} e_j$.
- The matrix $(A_{jk})_{j,k=1,\ldots,n}$ determines A in the given basis $(e_1, \ldots, e_n)$.

The composition of two linear maps $A : V \to W$ and $B : W \to Z$ is again a linear map $BA : V \to Z$.

Examples of Linear Maps

1. The identity $I : V \to V$ given by $Iv = v$ is linear. Its matrix in any basis is the Kronecker delta $\delta_{ij} = 1$ if $i = j$, 0 otherwise.
2. Let $C^1([a, b])$ be the space of continuously differentiable real-valued functions of $[a, b]$. Then the derivative $d/dx : C^1([a, b]) \to C([a, b])$ is a linear map.
3. Solving a differential equation $d/dxu(x) = f$ consists of finding the preimage of f under this linear map. This is the view of PDE from functional analysis.
4. Similarly, integration $f \mapsto \int_a^b f$ defines a linear function $C([a, b]) \to \mathbb{R}$.

Hom Spaces

Let $\mathrm{Hom}(V, W)$ be the set of all linear maps $V \to W$. Then $\mathrm{Hom}(V, W)$ is a vector space: If $\lambda \in k$, $v \in V$, and $A, B \in \mathrm{Hom}(V, W)$, define

$$(\lambda A)v = \lambda(Av)$$
$$(A + B)v = Av + Bv$$

- You can check that $\mathrm{Hom}(V, W)$ is a vector space.
- We can also define $\mathrm{End}(V) := Hom(V, V)$.
- Fact: $\mathrm{End}(k) \cong k$.
- Later, we will see that $\mathrm{Hom}(V, W) \cong V^* \otimes W$, where V^* is the dual space.

Part IV

ISOMORPHISMS

ISOMORPHISMS

Definition 0.7

Two vector spaces are ***isomorphic*** *if there exists a bijective linear map between them.*

- If $A : V \to W$ is an isomorphism and $(e_1, \ldots, e_n)$ is a basis for V, then $(Ae_1, \ldots, Ae_n)$ is a basis for W.

Coupled with the fact that a basis for V is equivalent to an isomorphism $V \to k^n$, we get:

Theorem 3

Any vector space V of dimension n is isomorphic to the space k^n of n-tuples.

Note that this isomorphism depends on the choice of basis! It is not *canonical*:

Definition 0.8

A linear map $V \to W$ is ***canonically defined*** *or* ***canonical*** *if it's definition does not depend on the basis chosen.*
V and W are ***canonically isomorphic*** *if there is a canonically defined isomorphism between the two.*

Canonical Isomorphisms

- We can *construct* an isomorphism by choosing a basis, but to show its canonical we must show that it gives the same values for any other choice of basis.
- As a general philosophy, canonically isomorphic vector spaces can be "identified" for all intents and purposes, while noncanonically isomorphic ones cannot be. Be careful though!

1. V is canonically isomorphic to itself via the identity map. More generally, any λI for $0 \neq \lambda \in k$, gives a canonical isomorphism.
2. If V is 1-dimensional, the isomorphism $\mathrm{End}(V) \to k$ is constructed by noting every element of $\mathrm{End}(V)$ is multiplication by a scalar. But this is not canonical: it requires a choice of vector $0 \neq v \in V$ that is mapped on $1 \in k$.

Part V

Making New Vector Spaces From Old Ones

Sums and Products

Let I be a (possibly infinite) set indexing a collection of vector spaces $\{V_i\}_{i \in I}$ defined over the same field k.

Definition 0.9

*The **direct sum** $\bigoplus_{i \in I} V_i$ is the set of tuples $(v_i)_{i \in I}$ with $v_i = 0$ for all but finitely many i.*
*The **direct product** $\prod_{i \in I} V_i$ is the set of all tuples $(v_i)_{i \in I}$.*

- We can define addition and scalar multiplication componenentwise for both: $\lambda (v_i)_{i \in I} = (\lambda v_i)_{i \in I}$ and $(v_i)_{i \in I} + (w_i)_{i \in I} = (v_i + w_i)_{i \in I}$. This makes them into vector spaces.
- Note that $\bigoplus_{i \in I} V_i \subset \prod_{i \in I} V_i$ and the two are the same if I is finite.

Exercise: Show that $\mathbb{R}^n \oplus \mathbb{R}^m$ is isomorphic to $\mathbb{R}^{n+m}$, but not canonically.

The Rank-Nullity Theorem

Theorem 4

Let $A : V \to W$ *be a linear transformation between finite-dimensional vector spaces. Then*

$$\dim \mathrm{im}\, A + \dim \ker A = \dim V$$

If you have taken abstract algebra, this theorem is just a consequence of the first isomorphism theorem: $V/\ker A \cong \mathrm{im}\, A$.

Part VI

The Dual Space and Hyperplanes

Dual Space

Definition 0.10

If V is a vector space, the dual vector space V^* is defined to be $\mathrm{Hom}(V, k)$.
*In other words, it is the set of all linear maps $V \to k$ (called **linear functionals**).*

1. Integration on $C([a, b])$ defines a linear functional. So does $\frac{d}{dx}|_{x=a}$ on the space of differentiable functions.
2. If $\mathbb{R}^2$ has coordinates $v = (x, y)$, then linear functionals are $f(v) = x - y$, $g(v) = 2x$.

Dual Basis

We now show that there exists an isomorphism $V \to V^*$

- Choose a basis $(e_1, \ldots, e_n)$ of V. We claim that the tuple $(e^1, \ldots, e^n)$ is a basis of V^*, called the **dual basis**, where we characterize e^i by

$$e^i(e_j) = \delta_{ij}$$

- They span the space: We have for $f \in V^*$ and $v \in V$ with $v = \sum_{k=1}^n v_k e_k$,

$$f(v) = f(\sum_{k=1}^n v_k e_k) = \sum_{k=1}^n v_k f(e_k) = \sum_{k=1}^n e^k(v) f(e_k)$$

  so $f = \sum_{k=1}^n e^k f(e_k)$.
- They are linearly independent: If $\sum_{k=1}^n \lambda_k e^k = 0$, then acting on e_j we get

$$0 = (\sum_{k=1}^n \lambda_k e^k)(e_j) = \lambda_j$$

  so all the λ_j are zero.

Dual Basis

How does this look explicitly?

- Pick a basis and view elements of V as column vectors. Let's do this for $\mathbb{R}^3$ with the standard basis, so

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Then the dual basis e^1, e^2, e^3 are can be viewed as row vectors:

$$e_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, v = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$$

  since the usual matrix multiplication gives $e^i \cdot e_j = \delta_{ij}$
- So, after picking a basis, the isomorphism $V \to V^*$ is given by $v \mapsto v^T$, the transpose!
- This isomorphism is **not** canonical.

A Bit of Geometry

We are familiar with planes, such as the set $\{x = 0\} \subset \mathbb{R}^3$ or {x+2y-z=0} $\subset \mathbb{R}^3$.

Definition 0.11

*The **hyperplane** annihilated by* $f \in V^*$ *is the set* $\{x \in V : f(v) = 0\} = \ker f$.

Theorem 5

The hyperplane annihilated by nonzero $f \in V^*$ *has dimension* $n - 1$. *(*$n = \dim V$*).*

Proof.

- The image of f either has dimension 0 or 1. Since it is nonzero, $\dim \mathrm{im}\, f = 1$.
- From the rank-nullity theorem, $\dim V = \dim \mathrm{im}\, f + \dim \ker f = 1 + \dim \ker f$. So $\dim \ker f = n - 1$, the dimension of the hyperplane.

□

As an example, the linear functional corresponding to $[a_1, a_2, a_3]$ in the standard basis is the hyperplane $\{a_1 x + a_2 y + a_3 z = 0\}$.

This allows us to generalize our intuition to higher dimensions!

Exercise Let $f_1, \ldots, f_m \in V^*$. Show that $\{v \in V : f_i(v), i = 1, \ldots, m\}$ is a linear subspace of V. Show that if $f_1, \ldots, f_m$ are linearly independent, then the dimension of that subspace is $n - m$ (where $n = \dim V$).

Exercises

1. Show that $\mathbb{R}^n \oplus \mathbb{R}^m$ is isomorphic to $\mathbb{R}^{n+m}$, but not canonically.
2. If V is one-dimensional, show that $\mathrm{End}(V)$ is isomorphic to k, but not canonically.
3. Go on wikipedia and look up the universal properties of $\oplus$ and $\times$. Verify that they are true.
4. Construct a **canonical** isomorphism between V and its double dual V^{**}, for V finite-dimensional.
5. Let V be the vector space of polynomials in x of degree $\leq d$ with coefficients in $\mathbb{R}$. Let $(1, x, x^2, x^3, \ldots, x^3)$ be the basis of V. For notational convenience, set $e_i = x^i$. Express the corresponding dual basis e^i in terms of the (higher) derivative operator $\frac{d^i}{dx^i}|_{x=0}$
6. Let $f_1, \ldots, f_m \in V^*$. Show that $\{v \in V : f_i(v), i = 1, \ldots, m\}$ is a linear subspace of V. Show that if $f_1, \ldots, f_m$ are linearly independent, then the dimension of that subspace is $n - m$ (where $n = \dim V$).